Math 245B Lecture 26 Notes

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1 Riesz Represetntation for Hilbert Spaces and Orthonormality

1.1 Riesz representation for Hilbert spaces

Let's finish up a proof from last time.

Theorem 1.1. *Let H be a Hilbert space, and let M be a closed subspace. Then any $x \in H$ can be written uniquely as $x = y + z$, where $y \in M$ and $z \in M^\perp$. We write $H = M \oplus M^\perp$.*

Proof. Let $\delta = \inf\{\|x - y\| : y \in M\}$. Pick $(y_n)_n$ in M such that $\|x - y_n\| \to \delta$. We claim that (y_n) is Cauchy. We have

$$\|y_n - y_m\|^2 + \|y_n + y_m - 2x\|^2 = 2(\|y_n - x\|^2 + \|y_m - x\|^2).$$

Rewrite this as

$$\|y_n - y_m\|^2 + 4\underbrace{\left\|\frac{y_n + y_m}{2} - x\right\|^2}_{\to\delta^2} = 2(\underbrace{\|y_n - x\|^2}_{\to\delta^2} + \underbrace{\|y_m - x\|^2}_{\to\delta^2}).$$

This is only possible if $\|y_n - y_m\| \to 0$. So the limit $y = \lim_n y_n$ exists. Moreover, $\|y - x\| = \delta$. This point is unique; if we had y, y' with the same property, the same identity above gives $\|y - y'\| = 0$.

It now remains to show that $z = x - y \in M^\perp$. Suppose not, and choose $v \in M$ such that $|\langle z, v\rangle| \in (0, \infty)$. Now consider

$$\|x - (y + tv)\|^2 = \underbrace{\|z\|^2}_{=\delta^2} + t^2\|v\|^2 - 2t\operatorname{Re}(\{z, v\}).$$

This can be made $< \delta^2$ unless $\langle z, v\rangle = 0$. $\square$

The first part of this proof is appealing to a particular property which does not only hold just in Hilbert spaces.

Definition 1.1. A Banach space $(\mathcal{X}, \|\cdot\|)$ is **uniformly convex** if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $x, y \in \mathcal{X}$ with $\|x\| = \|y\| = 1$, if $\|(x+y)/2\| > 1 - \delta$, then $\|x - y\| < \varepsilon$.

Example 1.1. For $1 < p < \infty$, $L^p(\mathbb{R})$ is uniformly convex.

Theorem 1.2 (Riesz[1])**.** *For any $f \in H^*$, there exists $y \in H$ such that $f(x) = \langle x, y\rangle$ and $\|f\| = \|y\|$.*

Proof. Assume $f \neq 0$, and let $M = \{x : f(x) = 0\}$. This is a closed, proper subspace of H. By the previous theorem, there must exist a $z \in H$ such that z is orthogonal to M. So pick $z \in M^\perp$ with $\|z\| = 1$. For any $x \in H$, consider $u = f(x)z - f(z)x$, which lies in M. So

$$0 = \langle u, z\rangle = f(x) \cdot 1 - f(z) \langle x, z\rangle .$$

That is, $f(x) = f(z) \langle x, z\rangle = \langle x, \overline{f(z)}z\rangle = \langle x, y\rangle$. □

Corollary 1.1. *Hilbert spaces are reflexive.*

1.2 Orthonormality

Definition 1.2. Let $(u_\alpha)_{\alpha \in H}$ be a collection of vectors in H. The collection is **orthonormal** if $\|u_\alpha\| = 1$ for all α, and when $\alpha \neq \beta$, $\langle u_\alpha, u_\beta\rangle = 0$.

Proposition 1.1 (Bessel's inequality)**.** *If $(u_\alpha)_\alpha$ is orthonormal in H, then*

$$\sum_\alpha |\langle x, u_\alpha\rangle|^2 \leq \|x\|.$$

Remark 1.1. When we are dealing with an uncountable set of vectors, we mean that all but countably many of them are orthonormal to x, so the sum makes sense.

Proof. Suppose $F \subseteq A$ and $|F| < \infty$. Then

$$\begin{aligned} 0 \leq \left\| x - \sum_{\alpha \in F} \langle x, u_\alpha\rangle u_\alpha \right\| &= \|x\|^2 - 2 \operatorname{Re} \left\langle x, \sum_{\alpha \in F} \langle x, u_\alpha\rangle u_\alpha \right\rangle + \sum_{\alpha \in F} |\langle x, u_\alpha\rangle|^2 \\ &= \|x\|^2 - 2 \sum_{\alpha \in F} |\langle x, u_\alpha\rangle|^2 + \sum_{\alpha \in F} |\langle x, u_\alpha\rangle|^2 \\ &= \|x\|^2 - \sum_{\alpha \in F} |\langle x, u_\alpha\rangle|^2. \end{aligned}$$

□

Theorem 1.3. *Let $(u_\alpha)_\alpha$ be an orthonormal set in H. The following are equivalent:*

[1] This is yet another theorem called the Riesz representation theorem.

1. *(completeness[2]) If $\langle x, u_\alpha \rangle = 0$ for all α, then $x = 0$.*

2. *(Parseval's identity) Besel's inequality is an equality for all x.*

3. *For all $x \in H$, we have $x = \sum_{\alpha \in A} \langle x, u_\alpha \rangle u_\alpha$.*

Remark 1.2. It is possible for the sum of the lengths in (3) to be infinite, so this sum is not absolutely convergent. But the sum of the squares must be finite, as shown by part (b).

Proof. (1) $\implies$ (3). Pick x. Bessel's inequality gives $\|x\|^2 \geq \sum_\alpha |\langle x, u_\alpha \rangle|^2$. So there are only countably many nonzero terms. Enumerate them as $\alpha_1, \alpha_2, \dots$. Consider $\sum_{i=1}^n \langle x, u_{\alpha_i} \rangle u_{\alpha_i}$. If $m > n$,

$$\left\| \sum_{i=n+1}^m \langle x, u_{\alpha_i} \rangle u_{\alpha_i} \right\| = \sum_{i=n+1}^m |\langle x, u_\alpha \rangle|^2 \xrightarrow{n,m \to \infty} 0.$$

So $\sum_{i=1}^n \langle x, u_{\alpha_i} \rangle u_{\alpha_i}$ is a Cauchy sequence, so it converges to some y. Now $y = x$ because for all α,

$$\langle y, \alpha \rangle = \begin{cases} \langle x, u_\alpha \rangle & \alpha = \alpha_i \text{ for some } i \\ 0 = \langle x, u_\alpha \rangle & \alpha \notin \{\alpha_1, \alpha_2, \dots\}. \end{cases}$$

This implies $y = x$ by (a).

(3) $\implies$ (2): Look that

$$\left\| x - \sum_{i=1}^n \langle x, u_{\alpha_i} \rangle u_{\alpha_i} \right\|.$$

This is the gap we found in Bessel's inequality. So we get Parseval in the limit as $n \to \infty$.

(2) $\implies$ (1): If $\|x\|^2 = \sum_\alpha |\langle x, u_\alpha \rangle|^2$, and the left hand side is nonzero, then there exists α such that $\langle x, u_\alpha \rangle \neq 0$. ☐

Definition 1.3. Any orthonormal set satisfying the previous theorem is a **basis**.

Theorem 1.4. *Any Hilbert space H has an orthonormal basis.*

Proof. Use Zorn's lemma. ☐

Remark 1.3. A basis is also a maximal orthonormal set.

Proposition 1.2. *H is separable if and only if it has a countable basis.*

Theorem 1.5. *Let H be a Hilbert space over $\mathbb{C}$.*

1. *If $\dim(H) = n < \infty$, then $H \cong \mathbb{C}^n$.*

[2] Add this to the list of the most overused words in mathematics.

2. *If* $\dim(H) = \infty$ *and* H *is separable, then* $H \cong \ell^2(\mathbb{N})$.

Proof. Suppose $\dim(H) = \infty$. Pick a basis $\{u_1, u_2, u_3, \dots\}$. For each $x \in H$, map $x \mapsto (\langle x, u_1 \rangle, \langle x, u_2 \rangle, \dots) \in \ell^2$. Parseval's identity says exactly that this is a unitary equivalence. $\square$